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Computable Legendrian invariants

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Abstract

We establish tools to facilitate the computation and application of the Chekanov–Eliashberg differential graded algebra (DGA), a Legendrian-isotopy invariant of Legendrian knots and links in standard contact three space. More specifically, we reformulate the DGA in terms of front projections, and introduce the characteristic algebra, a new invariant derived from the DGA. We use our techniques to distinguish between several previously indistinguishable Legendrian knots and links. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A *Legendrian knot* in standard contact \mathbb{R}^3 is a knot which is everywhere tangent to the two-plane distribution induced by the contact one-form $dz - ydx$. Two Legendrian knots are *Legendrian isotopic* if there is a smooth isotopy between them through Legendrian knots.

Broadly speaking, we wish to determine when two Legendrian knots are Legendrian isotopic. There are two “classical” invariants for knots under Legendrian isotopy, Thurston–Bennequin number tb and rotation number r . These form a complete set of invariants for some knots, including the unknot [3], torus knots [6], and the figure eight knot [6].

However, there do exist nonisotopic Legendrian knots of the same topological type with the same tb and r . The method for demonstrating this fact is a new, nonclassical invariant independently introduced by Chekanov [1] and Eliashberg [2]. We will use Chekanov’s combinatorial formulation of this invariant, which we call the Chekanov–Eliashberg differential graded algebra (DGA). Chekanov

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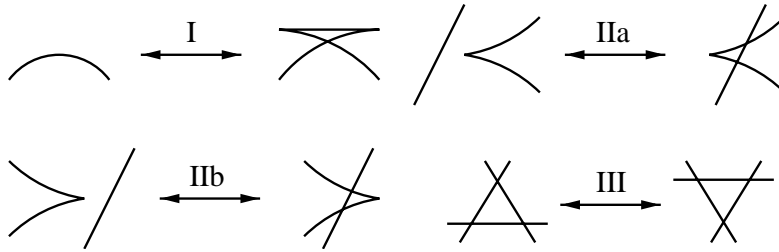


Fig. 1. The Legendrian Reidemeister moves which relate Legendrian-isotopic fronts. Reflections of these moves in the horizontal axis are also allowed.

introduced a concept of equivalence between DGAs which he called stable tame isomorphism; then two Legendrian-isotopic knots have equivalent DGAs.

The Chekanov–Eliashberg DGA was originally defined as an algebra over $\mathbb{Z}/2$ with grading over $\mathbb{Z}/(2r(K))$. This has subsequently been lifted [7] to an algebra over the ring $\mathbb{Z}[t, t^{-1}]$ with grading over \mathbb{Z} , by following the picture from symplectic field theory [4]; this lifted algebra is what we will actually call the Chekanov–Eliashberg DGA.

There are two standard methods to portray Legendrian knots in standard contact \mathbb{R}^3 , via projections to \mathbb{R}^2 : the *Lagrangian projection* to the xy plane, and the *front projection* to the xz plane. Chekanov and Eliashberg, motivated by the general framework of contact homology [2] and symplectic field theory [4], used the Lagrangian projection in their setups.

When we attempt to apply the Chekanov–Eliashberg DGA to distinguish between Legendrian knots, we encounter two problems. The first is that it is not easy to manipulate Lagrangian-projected knots. Chekanov gives a criterion in [1] for a knot diagram in \mathbb{R}^2 to be the Lagrangian projection of a Legendrian knot, but it remains highly nontrivial to determine by inspection when two Lagrangian projections represent Legendrian-isotopic knots.

For questions of Legendrian isotopy, the front projection is more convenient, because we know precisely what diagrams represent front projections of Legendrian knots. A *front* in \mathbb{R}^2 is simply a (continuous) embedding of S^1 into the xz plane, with a unique nonvertical tangent line at each point in the image (i.e., so that dz/dx exists at each point), except of course at crossings. Every front is the front projection of a Legendrian knot, and every Legendrian knot projects to such a front. In addition, Legendrian-isotopic fronts are always related by a series of Legendrian Reidemeister moves [13]; see Fig. 1.

The second problem is that it is difficult in general to tell when two DGAs are equivalent. To each DGA, Chekanov [1] associated an easy-to-compute Poincaré-type polynomial, which is invariant under DGA equivalence, and used this to exhibit two 5_2 knots which have the same classical invariants but are not Legendrian isotopic. On the other hand, the polynomial is only defined for Legendrian knots possessing so-called augmentations; in addition, there are often many nonisotopic knots with identical classical invariants and polynomial.

This paper develops techniques designed to address these problems. In Section 2, we reformulate the Chekanov–Eliashberg DGA for front projections, and discuss how it can often be easier to compute than the Lagrangian-projection version. (The case of multi-component links, for which the results of this paper hold with minor modifications, is explicitly addressed in Section 2.5.) In Section 3,

we introduce a new invariant, the characteristic algebra, which is derived from the DGA and is relatively easy to compute. The characteristic algebra is quite effective in distinguishing between Legendrian isotopy classes; it also encodes the information from both the Poincaré–Chekanov polynomial and a similar higher-order invariant. Section 4 applies the characteristic algebra to a number of knots and links which were previously indistinguishable.

2. Chekanov–Eliashberg DGA in the front projection

This section is devoted to a reformulation of the Chekanov–Eliashberg DGA from the Lagrangian projection to the more useful front projection. In Section 2.1, we introduce resolution, the technique used to translate from front projections to Lagrangian projections. We then define the DGA for the front of a knot in Section 2.2, and discuss a particularly nice and useful case in Section 2.3. In Section 2.4, we review the main results concerning the DGA from [1,7]. Section 2.5 discusses the adjustments that need to be made for multi-component links.

2.1. Resolution of a front

Given a front, we can find a Lagrangian projection which represents the same knot through the following construction, which is also considered in [8] under the name “morsification”.

Definition 2.1. The *resolution* of a front is the knot diagram obtained by resolving each of the singularities in the front as shown in Fig. 2.

The usefulness of this construction is shown by the following result, which implies that resolution is a map from front projections to Lagrangian projections which preserves Legendrian isotopy.

Proposition 2.2. *The resolution of the front projection of any Legendrian knot K is the Lagrangian projection of a knot Legendrian isotopic to K .*

A similar result also holds for multi-component links. Note that Proposition 2.2 is a bit stronger than the assertion from [8] that the regular isotopy type of the resolution is invariant under Legendrian isotopy of the front.

Proof. It suffices to distort the front K smoothly to a front K' so that the resolution of K is the Lagrangian projection of the knot corresponding to K' .

We choose K' to have the following properties; see Fig. 3 for an illustration. Suppose that there are at most k points in K with any given x coordinate. Outside of arbitrarily small “exceptional segments,” K' consists of straight line segments. These line segments each have slope equal to

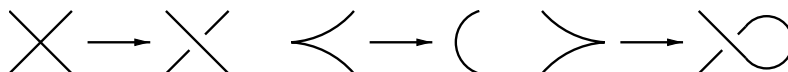


Fig. 2. Resolving a front into the Lagrangian projection of a knot.

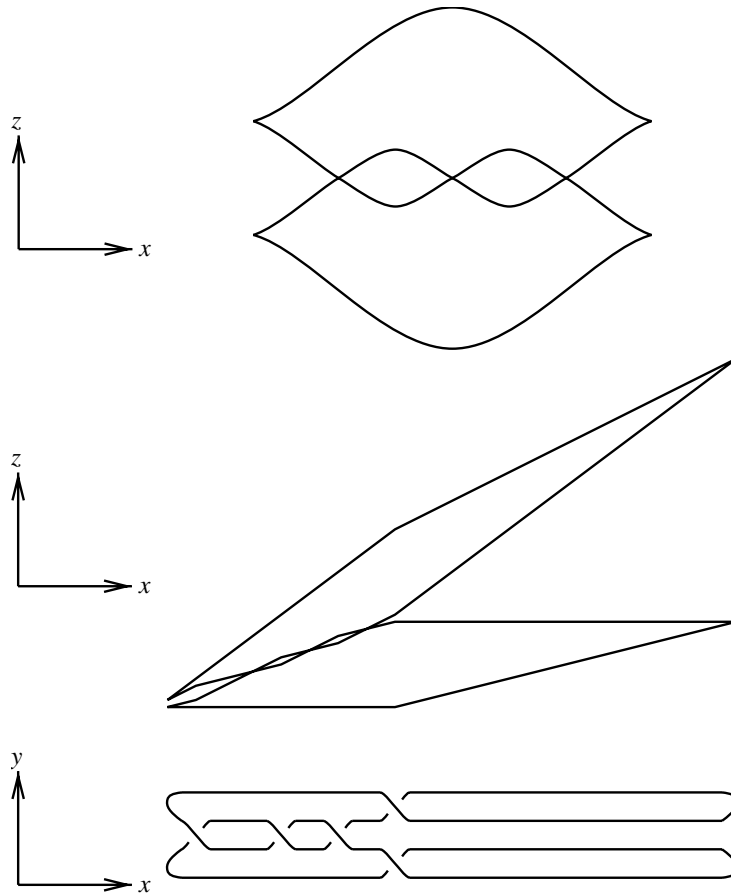


Fig. 3. A front projection for the left-handed trefoil (top) is distorted (middle) so that the corresponding Lagrangian projection (bottom), given by $y = dz/dx$, with the same x -axis as the middle diagram, is the resolution of the original front. The exceptional segments in the middle diagram appear as corners.

some integer between 0 and $k - 1$ inclusive; outside of the exceptional segments, for any given x coordinate, the slopes of the line segments at points with that x coordinate are all distinct. The purpose of the exceptional segments is to allow the line segments to change slopes, by interpolating between two slopes. When two line segments exchange slopes via exceptional segments, the line segment with higher z coordinate has higher slope to the left of the exceptional segment, and lower slope to the right.

It is always possible to construct such a distortion K' . Build K' starting from the left; a left cusp is simply two line segments of slope j and $j + 1$ for some j , smoothly joined together by appending an exceptional segment to one of the line segments. Whenever two segments need to cross, force them to do so by interchanging their slopes (again, with exceptional segments added to preserve smoothness). To create a right cusp between two segments, interchange their slopes so that they cross, and then append an exceptional segment just before the crossing to preserve smoothness.

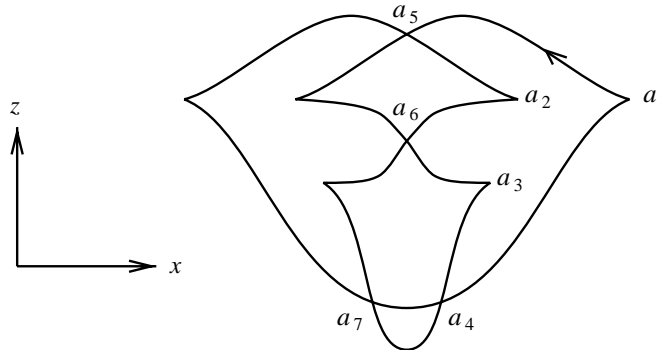


Fig. 4. The front projection of a figure eight knot, with vertices labelled.

We obtain the Lagrangian projection of the knot corresponding to K' by using the relation $y = dz/dx$. This projection consists of horizontal lines (parallel to the x -axis), outside of a number of crossings arising from the exceptional segments. These crossings can be naturally identified with the crossings and right cusps of K or K' . In particular, right cusps in K become the crossings associated to a simple loop. It follows that the Lagrangian projection corresponding to K' is indeed the resolution of K , as desired. \square

2.2. The DGA for fronts of knots

Suppose that we are given the front projection Y of an oriented Legendrian knot K . To define the Chekanov–Eliashberg DGA for Y , we simply examine the DGA for the resolution of Y and “translate” this in terms of Y . In the interests of readability, we will concentrate on describing the DGA solely in terms of Y , invoking the resolution only when the translation is not obvious.

The singularities of Y fall into three categories: crossings (nodes), left cusps, and right cusps. Ignore the left cusps, and call the crossings and right cusps *vertices*, with labels a_1, \dots, a_n (see Fig. 4); then the vertices of Y are in one-to-one correspondence with the crossings of the resolution of Y .

As an algebra, the Chekanov–Eliashberg DGA of the front Y is defined to be the free, noncommutative, unital algebra $A = \mathbb{Z}[t, t^{-1}] \langle a_1, \dots, a_n \rangle$ over $\mathbb{Z}[t, t^{-1}]$ generated by a_1, \dots, a_n . We wish to define a grading on A , and a differential ∂ on A which lowers the grading by 1.

We first address the grading of A . For an oriented path γ contained in the diagram Y , define $c(\gamma)$ to be the number of cusps traversed upwards, minus the number of cusps traversed downwards, along γ . Note that this is the opposite convention from the one used to calculate rotation number; if we consider Y itself to be an oriented closed curve, then $r(K) = -c(Y)/2$.

Let the degree of the indeterminate t be $2r(K)$. To grade A , it then suffices to define the degrees of the generators a_i ; we follow [7].

Definition 2.3. Given a vertex a_i , define the *capping path* γ_i , a path in Y beginning and ending at a_i , as follows. If a_i is a crossing, move initially along the segment of higher slope at a_i , in the direction of the orientation of Y ; then follow Y , not changing direction at any crossing, until a_i is reached

again. If a_i is a right cusp, then γ_i is the empty path, if the orientation of Y traverses a_i upwards, or the entirety of Y in the direction of its orientation, if the orientation of Y traverses a_i downwards.

Definition 2.4. If a_i is a crossing, then $\deg a_i = c(\gamma_i)$. If a_i is a right cusp, then $\deg a_i$ is 1 or $1 - 2r(K)$, depending on whether the orientation of Y traverses a_i upwards or downwards, respectively.

We thus obtain a grading for A over \mathbb{Z} . As an example, in the figure eight knot shown in Fig. 4, a_1, a_2, a_3, a_4 have degree 1, a_5, a_6 have degree 0, and a_7 has degree -1 .

It will be useful to introduce the sign function $\text{sgn } v = (-1)^{\deg v}$ on pure-degree elements of A , including vertices of Y ; note that any right cusp has negative sign. The Thurston–Bennequin number for K can be written as the difference between the numbers of positive-sign and negative-sign vertices in Y . Since $\deg t = 2r(K)$, the graded algebra A incorporates both classical Legendrian-isotopy invariants.

We next wish to define the differential ∂ on A . As in [1], we define ∂a_i for a generator a_i by considering a certain class of immersed disks in the diagram Y .

Definition 2.5. An *admissible map* on Y is an immersion from the two-disk D^2 to \mathbb{R}^2 which maps the boundary of D^2 into the knot projection Y , and which satisfies the following properties: the map is smooth except possibly at vertices and left cusps; the image of the map near any singularity looks locally like one of the diagrams in Fig. 5, excepting the two forbidden ones; and, in the notation of Fig. 5, there is precisely one initial vertex.

The singularities of an admissible map thus consist of one initial vertex, a number of corner vertices (possibly including some right cusps counted twice), and some other singularities which we will ignore. One type of corner vertex, the “downward” corner vertex as labelled in Fig. 5, will be important shortly in determining signs.

We remark that the allowed singularities in Definition 2.5 are derived by directly translating Chekanov’s formulation from the Lagrangian projection to the front projection. The forbidden singularities, however, warrant a bit of explanation. Call a point p in the domain of an admissible map, and its image under the map, *locally rightmost* if p attains a local maximum for the x coordinate of its image. Any locally rightmost point in the image of an admissible map must be the unique initial vertex of the map: this point must be a node or a right cusp, which cannot be a negative corner vertex (cf. Fig. 5). In particular, there must be a unique locally rightmost point in the image. Of the two forbidden singularities from Fig. 5, the left one is disallowed because the initial vertex is not rightmost, and the right one because there would be two locally rightmost points.

To each diffeomorphism class of admissible maps on Y , we will now associate a monomial in $\mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$. Let f be a representative of a diffeomorphism class, and suppose that f has corner vertices at $a_{j_1}, \dots, a_{j_\ell}$, counted twice where necessary, in counterclockwise order around the boundary of D^2 , starting just after the initial vertex, and ending just before reaching the initial vertex again. Then the monomial associated to f , and by extension to the diffeomorphism class of f , is

$$\alpha(f) = (\text{sgn } f) t^{-n(f)} a_{j_1} \dots a_{j_\ell},$$

where $(\text{sgn } f)$ is the parity ($+1$ for even, -1 for odd) of the number of downward corner vertices of f of even degree, and the winding number $n(f)$ is defined below.

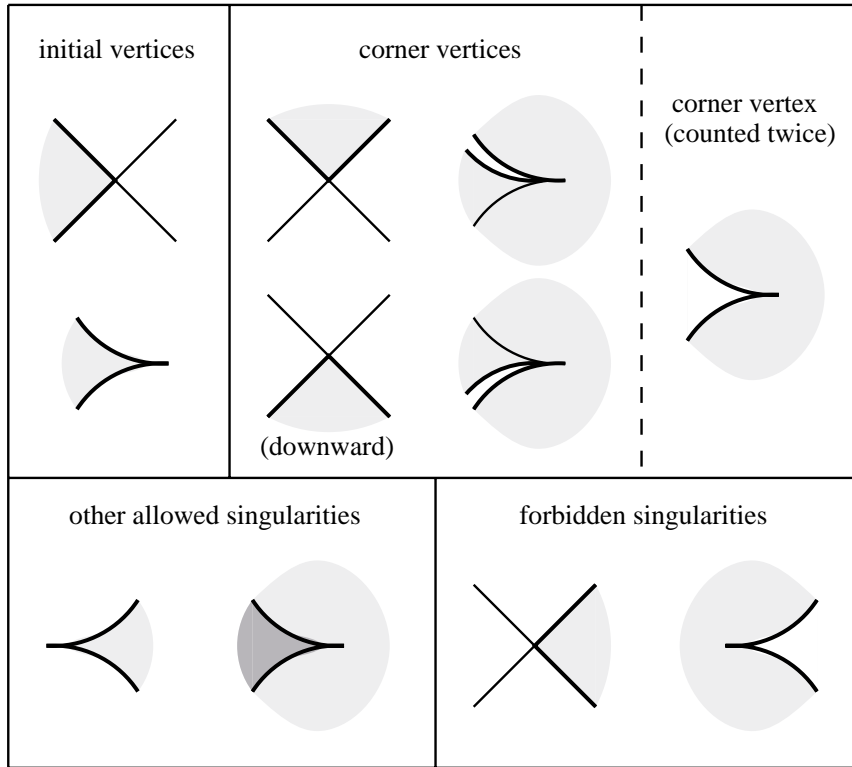


Fig. 5. Possible singularities in an admissible map, and their classification. The shaded area is the image of the map restricted to a neighborhood of the singularity; the heavy line indicates the image of the boundary of D^2 . In two of the diagrams, the heavy line has been shifted off of itself for clarity. The diagram with heavy shading indicates that the image overlaps itself.

The image $f(\partial D^2)$, oriented counterclockwise, lifts to a collection of oriented paths in the knot K . If a_i is the initial vertex of f , then the lift of $f(\partial D^2)$, along with the lifts of the capping paths $\gamma_i, -\gamma_{j_1}, \dots, -\gamma_{j_\ell}$, form a closed cycle in K . We then set $n(f)$ to be the winding number of this cycle around K , with respect to the orientation of K .

Definition 2.6. Given a generator a_i , we define

$$\partial a_i = \begin{cases} \sum \alpha(f) & \text{if } a_i \text{ is a crossing,} \\ 1 + \sum \alpha(f) & \text{if } a_i \text{ is a right cusp oriented upwards,} \\ t^{-1} + \sum \alpha(f) & \text{if } a_i \text{ is a right cusp oriented downwards,} \end{cases}$$

where the sum is over all diffeomorphism classes of admissible maps f with initial vertex at a_i . We extend the differential to the algebra A by setting $\partial(\mathbb{Z}[t, t^{-1}]) = 0$ and imposing the signed Leibniz rule $\partial(vw) = (\partial v)w + (\text{sgn } v)v(\partial w)$.

A few remarks are in order. The power of t in the definition of the monomial $\alpha(f)$ has been translated directly from the corresponding definition in [7]. It is easy to check that the signs also correspond to the signs in [7], after we replace a_i by $-a_i$ for each a_i which is “right-pointing”; that is, near which the knot is locally oriented from left to right for both strands.

Definition 2.6 depends on a choice of orientation of K . For an unoriented knot, we may similarly define the differential without the powers of t ; the DGA is then an algebra over \mathbb{Z} graded over $\mathbb{Z}/(2r(K))$, still a lifting of Chekanov’s original DGA over $\mathbb{Z}/2$.

As a final remark, if K is a stabilization, i.e., contains a local zigzag, then it is easy to see that there is an a_i such that $\partial a_i = 1$ or $\partial a_i = t^{-1}$. In this case, $\partial(a_j - a_i \partial a_j) = 0$ or $\partial(a_j - t a_i \partial a_j) = 0$ for all j , and the DGA collapses modulo tame isomorphisms (see Section 2.4). This was first noted in [1, Section 11.2].

For the front in Fig. 4, we may compute (somewhat laboriously) that

$$\partial a_1 = 1 + a_6 - t^2 a_6 a_4 a_6 a_7 - t^2 (1 - t a_6 a_5) a_3 a_6 a_7 + t a_6 a_2 (1 - t a_6 - t^2 a_7 a_4 a_6) a_7,$$

$$\partial a_2 = 1 - t a_5 a_6,$$

$$\partial a_3 = t^{-1} - a_6 - t a_6 a_7 a_4,$$

$$\partial a_4 = \partial a_5 = \partial a_6 = \partial a_7 = 0.$$

See Fig. 6 for a depiction of two of the admissible maps counted in ∂a_1 .

To illustrate the calculation of the sign and power of t associated to an admissible map, consider the term $t^3 a_6 a_5 a_3 a_6 a_7$ in ∂a_1 above. The sign of this term is $(-\text{sgn } a_5)(-\text{sgn } a_6) = +1$. To calculate the power of t , we count, with orientation, the number of times the cycle corresponding to this map passes through a_1 . The boundary of the immersed disk passes through a_1 , contributing 1; γ_1 trivially does not pass through a_1 , contributing 0; and $-\gamma_3, -\gamma_6, -\gamma_7$ pass through a_1 , while $-\gamma_5$ does not, contributing a total of -4 . It follows that the power of t is $t^{-(1+0-4)} = t^3$.

2.3. Simple fronts

Since the behavior of an admissible map near a right cusp can be complicated, our formulation of the differential algebra may seem no easier to compute than Chekanov’s. There is, however, one class of fronts for which the differential is particularly easy to compute.

Definition 2.7. A front is *simple* if it is smoothly isotopic to a front all of whose right cusps have the same x coordinate.

Any front can be Legendrian-isotoped to a simple front: “push” all of the right cusps to the right until they share the same x coordinate. (In the terminology of Fig. 1, a series of IIb moves can turn any front into a simple front.)

For a simple front, the boundary of any admissible map must begin at a node or right cusp (the initial vertex), travel leftwards to a left cusp, and then travel rightwards again to the initial vertex. Outside of the initial vertex and the left cusp, the boundary can only have very specific corner vertices: each corner vertex must be a crossing, and, in a neighborhood of each of these nodes, the

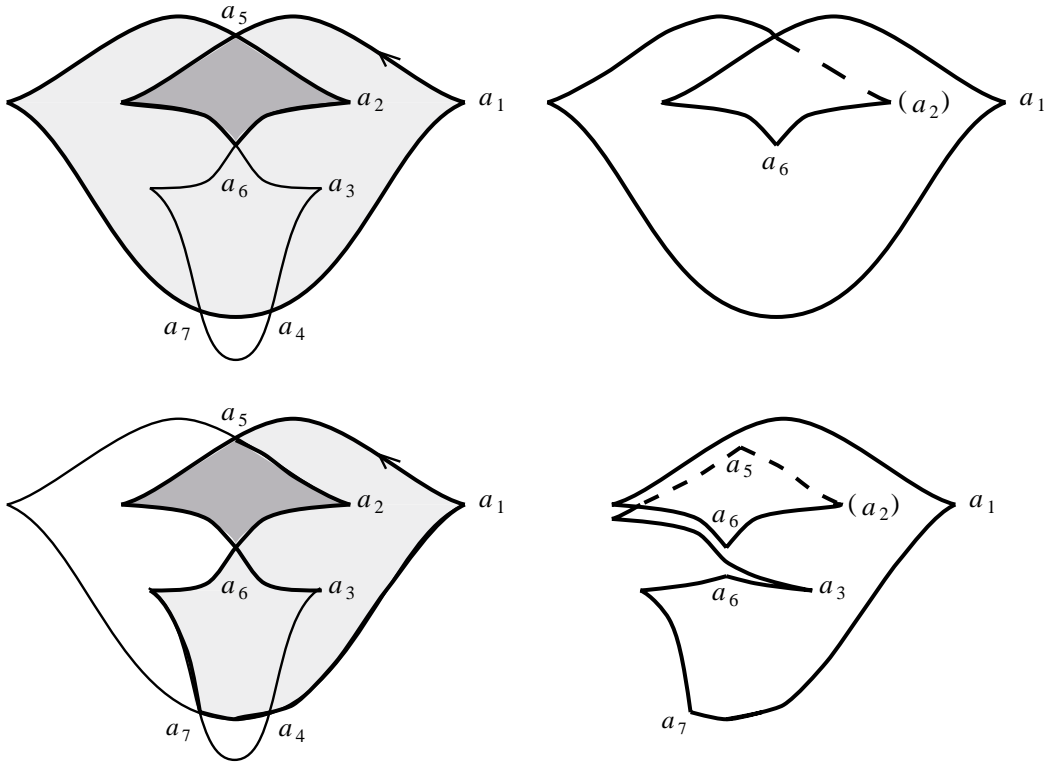


Fig. 6. The admissible maps which correspond to the terms a_6 (top) and $t^3 a_6 a_5 a_3 a_6 a_7$ (bottom) in ∂a_1 , for the front from Fig. 4. The heavy lines indicate the image of the boundary of D^2 ; the heavy shading indicates where the images overlap themselves. For clarity, the images of the maps are redrawn to the right.

image of the map must only occupy one of the four regions surrounding the crossing. In particular, the map is an embedding, not just an immersion.

Example 2.8. It is easy to calculate the differential for the simple-front version of the figure eight knot given in Fig. 7:

$$\begin{aligned} \partial a_1 &= 1 + a_6 + ta_{10}a_5, & \partial a_4 &= t^{-1} + a_8a_7 - a_9a_6 - ta_9a_{10}a_5, \\ \partial a_2 &= 1 - ta_9a_{10}, & \partial a_5 &= a_7 + a_{11} + ta_{11}a_8a_7, \\ \partial a_3 &= t^{-1} - a_{10} - ta_{10}a_{11}a_8, & \partial a_6 &= -ta_{10}a_7 - ta_{10}a_{11} - t^2a_{10}a_{11}a_8a_7, \\ \partial a_7 &= \partial a_8 = \partial a_9 = \partial a_{10} = \partial a_{11} = 0. \end{aligned}$$

For the signs, note that a_1, a_2, a_3, a_4 , and a_8 have degree 1, a_7 and a_{11} have degree -1 , and the other vertices have degree 0; for the powers of t , note that $\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_{10}$, and γ_{11} pass through a_1 , while the other capping paths do not.

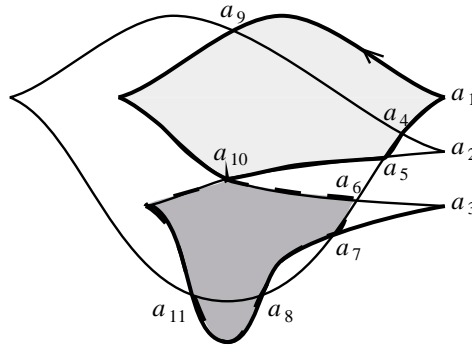


Fig. 7. A simple-front version of the front from Fig. 4, with two admissible maps drawn. The top shaded region corresponds to the term $ta_{10}a_5$ in ∂a_1 ; the bottom shaded region corresponds to the term $-ta_{10}a_7$ in ∂a_6 .

2.4. Properties of the DGA

In this section, we summarize the properties of the Chekanov–Eliashberg DGA. These results were originally proven over $\mathbb{Z}/2$ in [1], and then extended over $\mathbb{Z}[t, t^{-1}]$ in [7]. Proofs can be found in [7] in the Lagrangian-projection setup, or in [11] in the front-projection setup.

Proposition 2.9 (Chekanov [1], Etnyre et al. [7]). *For the DGA associated to a Legendrian knot, ∂ lowers degree by 1.*

Proposition 2.10 (Chekanov [1], Etnyre et al. [7]). *For the DGA associated to a Legendrian knot, $\partial^2 = 0$.*

To state the invariance result for the DGA, we need to recall several definitions from [1] or [7].

An (algebra) automorphism of a graded free algebra $\mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$ is *elementary* if it preserves grading and sends some a_i to $a_i + v$, where v does not involve a_i , and fixes the other generators $a_j, j \neq i$. A *tame automorphism* of $\mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$ is any composition of elementary automorphisms; a *tame isomorphism* between two free algebras $\mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$ and $\mathbb{Z}[t, t^{-1}]\langle b_1, \dots, b_n \rangle$ is a grading-preserving composition of a tame automorphism and the map sending a_i to b_i for all i . Two DGAs are then *tamely isomorphic* if there is a tame isomorphism between them which maps the differential on one to the differential on the other.

Let E be a DGA with generators e_1 and e_2 , such that $\partial e_1 = \pm e_2$, $\partial e_2 = 0$, both e_1 and e_2 have pure degree, and $\deg e_1 = \deg e_2 + 1$. Then an *algebraic stabilization* of a DGA $(A = \mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle, \partial)$ is a graded coproduct

$$(S(A), \partial) = (A, \partial) \amalg (E, \partial) = (\mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n, e_1, e_2 \rangle, \partial)$$

with differential and grading induced from A and E . Finally, two DGAs are *equivalent* if they are tamely isomorphic after some (possibly different) number of (possibly different) algebraic stabilizations of each.

We can now state the main invariance result.

Theorem 2.11 (Chekanov [1], Etnyre et al. [7]). *Fronts of Legendrian-isotopic knots have equivalent DGAs.*

Corollary 2.12 (Chekanov [1], Etnyre et al. [7]). *The graded homology of the DGA associated to a Legendrian knot is invariant under Legendrian isotopy.*

2.5. The DGA for fronts of links

In this section, we describe the modifications of the definition of the Chekanov–Eliashberg DGA necessary for Legendrian links in standard contact \mathbb{R}^3 . Here the DGA has an infinite family of gradings, as opposed to one, and is defined over a ring more complicated than $\mathbb{Z}[t, t^{-1}]$. The DGA for links also includes some information not found for knots.

Let L be an oriented Legendrian link, with components L_1, \dots, L_k ; in this section, for ease of notation, we will also use L, L_1, \dots, L_k to denote the corresponding front projections. Chekanov’s original definition [1] of the DGA for L gives an algebra over $\mathbb{Z}/2$ graded over $\mathbb{Z}/(2r(L))$, where $r(L) = \gcd(r(L_1), \dots, r(L_k))$; we will extend this to an algebra over $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ graded over \mathbb{Z} , and our set of gradings will be more refined than in [1]. We will also discuss an additional structure on the DGA introduced by Michatchev [10].

As in Section 2.2, let a_1, \dots, a_n be the vertices (crossings and right cusps) of L . We associate to L the algebra

$$A = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}] \langle a_1, \dots, a_n \rangle$$

with differential and grading to be defined below.

For each crossing a_i , let $N_u(a_i)$ and $N_l(a_i)$ denote neighborhoods of a_i on the two strands intersecting at a_i , so that the slope of $N_l(a_i)$ is greater than the slope of $N_u(a_i)$, i.e., $N_u(a_i)$ is lower than $N_l(a_i)$ in y coordinate. If a_i is a right cusp, define $N_u(a_i) = N_l(a_i)$ to be a neighborhood of a_i in L . For any vertex a_i , we may then define two numbers $u(a_i)$ and $l(a_i)$, the indices of the link components containing $N_u(a_i)$ and $N_l(a_i)$, respectively.

For each $j = 1, \dots, k$, fix a base point p_j on L_j , away from the singularities of L , so that L_j is oriented from left to right in a neighborhood of p_j . To a crossing a_i , we associate two capping paths γ_i^u and γ_i^l : γ_i^u is the path beginning at $p_{u(a_i)}$ and following $L_{u(a_i)}$ in the direction of its orientation until a_i is reached through $N_u(a_i)$; γ_i^l is the analogous path in $L_{l(a_i)}$ beginning at $p_{l(a_i)}$ and ending at a_i through $N_l(a_i)$. (If $u(a_i) = l(a_i)$, then one of γ_i^u and γ_i^l will contain the other.) Note that, by this definition, when a_i is a right cusp, γ_i^u and γ_i^l are both the path beginning at $p_{u(a_i)} = p_{l(a_i)}$ and ending at a_i .

Definition 2.13. For $(\rho_1, \dots, \rho_{k-1}) \in \mathbb{Z}^{k-1}$, we may define a \mathbb{Z} grading on A by

$$\deg a_i = \begin{cases} 1 & \text{if } a_i \text{ is a right cusp,} \\ c(\gamma_i^u) - c(\gamma_i^l) + 2\rho_{u(a_i)} - 2\rho_{l(a_i)} & \text{if } a_i \text{ is a crossing,} \end{cases}$$

where we set $\rho_k = 0$. We will only consider gradings on A obtained in this way.

The set of gradings on A is then indexed by \mathbb{Z}^{k-1} . (In particular, a knot has precisely one grading, the one given in Section 2.2). Our motivation for including precisely this set of gradings is given by the following easily proven observation.

Lemma 2.14. *The collection of possible gradings on A is independent of the choices of the points p_j .*

If a_i is contained entirely in component L_j , then the degree of a_i may differ from how we defined it in Definition 2.4 with L_j a knot by itself. It is easy to calculate that the difference between the two degrees will always be either 0 or $2r(L_j)$.

We may define the sign function on vertices, as usual, by $\text{sgn } a_i = (-1)^{\deg a_i}$. This is well-defined and independent of the choice of grading: $\text{sgn } a_i = -1$ if a_i is a right cusp; $\text{sgn } a_i = 1$ if a_i is a crossing with both strands pointed in the same direction (either both to the left or both to the right); and $\text{sgn } a_i = -1$ if a_i is a crossing with strands pointed in opposite directions. Note that $tb(L) = \sum_{i=1}^n \text{sgn } a_i$.

We may still define the differential of a generator a_i as in Definition 2.6, but we must now redefine $\alpha(f)$ for an admissible map f . Suppose that f has initial vertex a_i and corner vertices a_{i_1}, \dots, a_{i_m} . Then the lift of $f(\partial D^2)$ to L , together with the lifts of $\gamma_i^u, -\gamma_{i_1}^u, \dots, -\gamma_{i_m}^u, -\gamma_i^l, \gamma_{i_1}^l, \dots, \gamma_{i_m}^l$, form a closed cycle in L . Let the winding number of this cycle around component L_j be $n_j(f)$. Also, define $\text{sgn } f$, as before, to be the parity of the number of downward corner vertices of f with positive sign.

We now set

$$\alpha(f) = (\text{sgn } f) t_1^{-n_1(f)} \cdots t_k^{-n_k(f)} a_{i_1} \cdots a_{i_m}.$$

The differential ∂ can then be defined on A essentially as in Definition 2.6, except that we now have $\partial(\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]) = 0$, and

$$\partial a_i = \begin{cases} \sum \alpha(f) & \text{if } a_i \text{ is a crossing,} \\ 1 + \sum \alpha(f) & \text{if } a_i \text{ is a right cusp.} \end{cases}$$

Note that the signed Leibniz rule does not depend on the choice of base points p_j , since the signs ($\text{sgn } a_i$) are independent of this choice. Also, because of a different choice of capping paths, we always add 1 to a right cusp; cf. Definition 2.6.

In practice, there is a simple way to calculate $n_j(f)$: it is the signed number of times $f(\partial D^2)$ crosses p_j . Indeed, the winding number of the appropriate cycle around L_j is the signed number of times that it crosses a point on L_j just to the left of p_j . No capping path γ_i^u or γ_i^l , however, crosses this point. Hence $n_j(f)$ counts the number of times $f(\partial D^2)$ crosses a point just to the left of p_j ; we could just as well consider p_j instead of this point.

We next examine the effect of changing the base points p_j on the differential ∂ . Consider another set of base points \tilde{p}_j , giving rise to capping paths $\tilde{\gamma}_i^u, \tilde{\gamma}_i^l$, and let ξ_j be the oriented path in L_j from \tilde{p}_j to p_j . Then

$$\tilde{\gamma}_i^u - \gamma_i^u = \begin{cases} \xi_{u(a_i)}, & N_u(a_i) \subset \xi_{u(a_i)}, \\ \xi_{u(a_i)} - L_{u(a_i)}, & N_u(a_i) \not\subset \xi_{u(a_i)} \end{cases}$$

and similarly for $\tilde{\gamma}_i^l - \gamma_i^l$. We conclude the following result:

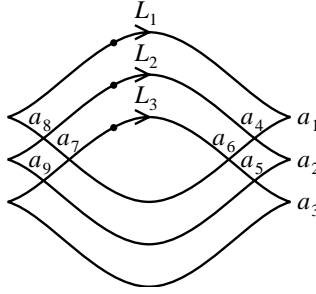


Fig. 8. An oriented link L with components L_1 , L_2 , and L_3 , with corresponding base points p_1 , p_2 , and p_3 marked but not labelled.

Lemma 2.15. *The differential on A , calculated with base points \tilde{p}_j , is related to the differential calculated with p_j , by intertwining with the following automorphism on A :*

$$a_i \mapsto \begin{cases} a_i, & N_u(a_i) \subset \xi_{u(a_i)} \text{ and } N_l(a_i) \subset \xi_{l(a_i)}, \\ t_{l(a_i)}^{-1} a_i, & N_u(a_i) \subset \xi_{u(a_i)} \text{ and } N_l(a_i) \not\subset \xi_{l(a_i)}, \\ t_{u(a_i)} a_i, & N_u(a_i) \not\subset \xi_{u(a_i)} \text{ and } N_l(a_i) \subset \xi_{l(a_i)}, \\ t_{u(a_i)} t_{l(a_i)}^{-1} a_i, & N_u(a_i) \not\subset \xi_{u(a_i)} \text{ and } N_l(a_i) \not\subset \xi_{l(a_i)}. \end{cases}$$

For example, consider the link L in Fig. 8, with base points as shown. To give a grading to the DGA on L , choose $(\rho_1, \rho_2) \in \mathbb{Z}^2$. The degrees of generators are as follows:

$$\begin{aligned} \deg a_1 &= 1, & \deg a_4 &= 1 + 2\rho_2 - 2\rho_1, & \deg a_7 &= -1 + 2\rho_1, \\ \deg a_2 &= 1, & \deg a_5 &= 1 - 2\rho_2, & \deg a_8 &= -1 + 2\rho_1 - 2\rho_2, \\ \deg a_3 &= 1, & \deg a_6 &= 1 - 2\rho_1, & \deg a_9 &= -1 + 2\rho_2. \end{aligned}$$

The differential ∂ is then given by

$$\begin{aligned} \partial a_1 &= 1 + t_1 + t_1 t_2^{-1} a_8 a_4 + t_1 t_3^{-1} a_7 a_6, & \partial a_4 &= t_2 t_3^{-1} a_9 a_6, & \partial a_7 &= a_8 a_9, \\ \partial a_2 &= 1 + t_2 + t_2 t_3^{-1} a_9 a_5 + a_4 a_8, & \partial a_5 &= a_6 a_8, & \partial a_8 &= 0, \\ \partial a_3 &= 1 + t_3 + a_5 a_9 + a_6 a_7, & \partial a_6 &= 0, & \partial a_9 &= 0. \end{aligned}$$

We can now state several properties of the link DGA, the analogues of the results for knots in Section 2.4.

Proposition 2.16. *If (A, ∂) is a DGA associated to the link L , then $\partial^2 = 0$, and ∂ lowers degree by 1 for any of the gradings of A .*

The main invariance result requires a slight tweaking of the definitions. Define elementary and tame automorphisms as in Section 2.4; now, however, let a tame isomorphism between algebras generated by a_1, \dots, a_n and b_1, \dots, b_n be a grading-preserving composition of a tame automorphism and a map sending a_i to $(\prod_{m=1}^k t_m^{v_{k,m}}) b_i$, for any set of integers $\{v_{k,m}\}$. (This definition is necessitated by Lemma 2.15.) Define algebraic stabilization and equivalence as before.

Proposition 2.17. *If L and L' are Legendrian-isotopic oriented links, then for any grading of the DGA for L , there is a grading of the DGA for L' so that the two DGAs are equivalent.*

The proofs of Propositions 2.16 and 2.17 will be omitted here, as they are simply variants on the proofs of Propositions 2.9 and 2.10 and Theorem 2.11; see [1] or [11].

Our set of gradings for A is more restrictive than the set of “admissible gradings” postulated in [1]. To see this, we first translate our criteria for gradings to the Lagrangian-projection picture, and then compare with Chekanov’s original criteria.

Consider a Legendrian link L with components L_1, \dots, L_k . By perturbing L slightly, we may assume that the crossings of $\pi_{xy}(L)$ are orthogonal, where π_{xy} is the projection map $(x, y, z) \mapsto (x, y)$; as usual, label these crossings a_1, \dots, a_n . Choose neighborhoods $N_u(a_i)$ and $N_l(a_i)$ in L of the two points mapping to a_i under π_{xy} , so that $N_u(a_i)$ lies above $N_l(a_i)$ in z coordinate, and let $u(a_i)$ and $l(a_i)$ be the indices of the link components on which these neighborhoods lie.

For each j , choose a point p_j on L_j , and let θ_j be an angle, measured counterclockwise, from the positive x -axis to the oriented tangent to L_j at p_j ; note that θ_j is only well-defined up to multiples of 2π . Let $r_u(a_i)$ be the counterclockwise rotation number (the number of revolutions made) for the path in $\pi(L_{u(a_i)})$ beginning at $p_{u(a_i)}$ and following the orientation of $L_{u(a_i)}$ until a_i is reached via $N_u(a_i)$; similarly define $r_l(a_i)$. Then the gradings for the DGA of L are given by choosing $(\rho_1, \dots, \rho_{k-1}) \in \mathbb{Z}^{k-1}$ and setting

$$\deg a_i = 2(r_l(a_i) - r_u(a_i)) + (\theta_{l(a_i)} - \theta_{u(a_i)})/\pi + 2\rho_{u(a_i)} - 2\rho_{l(a_i)} - 1/2.$$

By comparison, the allowed degrees in [1] are given by

$$\deg a_i = 2(r_l(a_i) - r_u(a_i)) + (\theta_{l(a_i)} - \theta_{u(a_i)})/\pi + \rho_{u(a_i)} - \rho_{l(a_i)} - 1/2.$$

The difference arises from the fact that Chekanov never uses the orientations of the link components; this forces $\theta_{l(a_i)}$ and $\theta_{r(a_i)}$ to be well-defined only up to integer multiples of π , rather than 2π .

We now discuss a refinement of the DGA for a link L , which is a minor variant of the relative homotopy splitting from [10]. The key observation, due to Michatchev, is that the information in the DGA can be split into pieces, because the link has several distinct components.

Definition 2.18. To any k -component Legendrian link L , we associate the *link DGA* (A^L, ∂) defined as follows. The algebra A^L is the free unital algebra over $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ generated by the vertices $\{a_i\}$ and orthogonal idempotents $\{e_j\}_{j=1}^k$, modulo the relations

- (1) $a_{i_1}a_{i_2} = 0$ if $l(a_{i_1}) \neq u(a_{i_2})$,
- (2) $e_j a_i = \delta_{u(a_i), j}$,
- (3) $a_i e_j = \delta_{l(a_i), j}$,
- (4) $e_{j_1} e_{j_2} = \delta_{j_1, j_2}$,

where δ is the Kronecker delta function. The differential ∂ on A^L is defined on generators as follows: $\partial e_j = 0$ for all j , and ∂a_i is the usual differential of a_i on A , except with any constant term replaced by the constant term times $e_{u(a_i)}$; as usual, we extend ∂ to A^L by the Leibniz rule. A grading on A^L is any grading inherited from A , after we set $\deg e_j = 0$.

Note that A^L is a familiar object from noncommutative algebra: it is simply the path algebra of a certain quiver. The vertices of the quiver are the k components of L , and the edges are the collection of arrows from $u(a_i)$ to $l(a_i)$, where a_i ranges over all vertices of the front of L .

The point is that all nonconstant monomials appearing in the usual definition of ∂a_i also appear in A^L : for such a monomial $a_{i_1} \dots a_{i_k}$, we have $u(a_{i_{p+1}}) = l(a_{i_p})$ for all p . In addition, we can decompose A^L into a direct sum $\oplus A_{j_1 j_2}^L$, where $A_{j_1 j_2}^L$ is generated additively by monomials $a_{i_1} \dots a_{i_k} \in A^L$ with $u(a_{i_1}) = j_1$ and $l(a_{i_k}) = j_2$, along with e_{j_1} if $j_1 = j_2$. Note that $a_i, \partial a_i \in A_{u(a_i)l(a_i)}^L$; we conclude that ∂ preserves each $A_{j_1 j_2}^L$.

We may define (grading-preserving) elementary and tame automorphisms and tame isomorphisms for link DGAs as for usual DGAs, with the additional stipulation that all maps must preserve the link structure by preserving $A_{j_1 j_2}^L$ for all j_1, j_2 . Similarly, we may define an algebraic stabilization of a link DGA, with the additional stipulation that the two added generators both belong to the same $A_{j_1 j_2}^L$. As usual, we then define two link DGAs to be equivalent if they are tamely isomorphic after some number of algebraic stabilizations. We omit the proof of the following result, which again is simply a variant on the proof of the corresponding result for knots.

Proposition 2.19. *If L and L' are Legendrian-isotopic oriented links, then for any grading of the link DGA for L , there is a grading of the link DGA for L' such that the two are equivalent.*

In this paper, we will not use the full strength of the link DGA. We will, however, apply first-order Poincaré–Chekanov polynomials derived from the link DGA; we now describe these polynomials, first mentioned in [10]. For the definition of augmentations for knots, and background on Poincaré–Chekanov polynomials, please refer to [1] or Section 3.2.

Assume that $r(L_1) = \dots = r(L_k) = 0$, and give A^L some fixed grading. We work over $\mathbb{Z}/2$; that is, set $t_j = 1$ for all j , and reduce modulo 2.

Definition 2.20. An *augmentation* of the link DGA A^L is an algebra map $\varepsilon: A^L \rightarrow \mathbb{Z}/2$ such that $\varepsilon \circ \partial = 0$, $\varepsilon(a_i) = 0$ when $u(a_i) \neq l(a_i)$, and $\varepsilon(e_j) = 1$ for all j . Note that an augmentation of the link DGA produces augmentations of the DGAs of each individual knot L_1, \dots, L_k , and conversely.

An augmentation ε , as usual, gives rise to a first-order Poincaré–Chekanov polynomial $P^{\varepsilon,1}(\lambda)$. This polynomial splits into k^2 polynomials $P_{j_1 j_2}^{\varepsilon,1}(\lambda)$, corresponding to the pieces in $A_{j_1 j_2}^L$; we now define these polynomials more carefully.

The polynomials $P_{jj}^{\varepsilon,1}(\lambda)$ are precisely the polynomials $P^{\varepsilon_j,1}(\lambda)$ for each individual link component L_j , where ε_j is the induced augmentation on the DGA of L_j . We can define $P_{j_1 j_2}^{\varepsilon,1}(\lambda)$ for $j_1 \neq j_2$ as follows. For $a_i \in A_{j_1 j_2}^L$, define $\partial_\varepsilon^{(1)} a_i$ to be the image of ∂a_i under the following operation: discard all terms in ∂a_i containing more than one a_m with $u(a_m) \neq l(a_m)$, and replace each a_m in ∂a_i by $\varepsilon(a_m)$ whenever $u(a_m) = l(a_m)$. If we write $V_{j_1 j_2}$ as the vector space over $\mathbb{Z}/2$ generated by $\{a_i \in A_{j_1 j_2}^L\}$, then $\partial_\varepsilon^{(1)}$ preserves $V_{j_1 j_2}$ and $(\partial_\varepsilon^{(1)})^2 = 0$. We may then set $P_{j_1 j_2}^{\varepsilon,1}(\lambda)$ to be the Poincaré polynomial of $\partial_\varepsilon^{(1)}$ on $V_{j_1 j_2}$, i.e., the polynomial in λ whose λ^i coefficient is the dimension of the i th graded piece of $(\ker \partial_\varepsilon^{(1)})/(\text{im } \partial_\varepsilon^{(1)})$.

We may also define higher-order Poincaré–Chekanov polynomials $P_{j_1 j_2}^{e,n}(\lambda)$ by examining the action of ∂ on $A_{j_1 j_2}^L$, but we will not need these here.

The following result, which follows directly from Proposition 2.19 and Chekanov’s corresponding result from [1], will be used in Section 4.

Theorem 2.21. *Suppose that L and L' are Legendrian-isotopic oriented links. Then, for any given grading and augmentation of the link DGA for L , there is a grading and augmentation of the link DGA for L' so that the first-order Poincaré–Chekanov polynomials $P_{j_1 j_2}^{e,1}$ for L and L' are equal for all j_1, j_2 .*

For unoriented links, we simply expand the set of allowed gradings $(\rho_1, \dots, \rho_{k-1})$ to allow half integers, as in [1]. Indeed, a grading of half integers $(\rho_1, \dots, \rho_{k-1})$ corresponds to changing the original orientation of L by either reversing the orientation of $\{L_j : 2\rho_j \text{ odd}\}$, or reversing the orientations of L_k and $\{L_j : 2\rho_j \text{ even}\}$. We deduce this by examining how the capping paths and degrees change when we change the orientation (and hence base point) of one link component L_j .

To conclude this section, we give an example. For the link from Fig. 8, an augmentation is any map with $\varepsilon(a_i) = 0$ for $i \geq 4$. Then $\partial_e^{(1)}$ is identically zero, and the first-order Poincaré–Chekanov polynomials simply measure the degrees of the a_i . More precisely, for a choice of grading $(\rho_1, \rho_2) \in \mathbb{Z}^2$, we have

$$\begin{aligned} P_{11}^{e,1}(\lambda) &= \lambda, & P_{21}^{e,1}(\lambda) &= \lambda^{1+2\rho_2-2\rho_1}, & P_{31}^{e,1}(\lambda) &= \lambda^{1-2\rho_1}, \\ P_{12}^{e,1}(\lambda) &= \lambda^{-1+2\rho_1-2\rho_2}, & P_{22}^{e,1}(\lambda) &= \lambda, & P_{32}^{e,1}(\lambda) &= \lambda^{1-2\rho_2}, \\ P_{13}^{e,1}(\lambda) &= \lambda^{-1+2\rho_1}, & P_{23}^{e,1}(\lambda) &= \lambda^{-1+2\rho_2}, & P_{33}^{e,1}(\lambda) &= \lambda. \end{aligned}$$

3. The characteristic algebra

We would like to use the Chekanov–Eliashberg DGA to distinguish between Legendrian isotopy classes of knots. Unfortunately, it is often hard to tell when two DGAs are equivalent. In particular, the homology of a DGA is generally infinite-dimensional and difficult to grasp; this prevents us from applying Corollary 2.12 directly.

Up to now, the only known “computable” Legendrian invariants—that is, nonclassical invariants which can be used in practice to distinguish between Legendrian isotopy classes of knots—were the first-order Poincaré–Chekanov polynomials. (Another invariant of Chekanov and Pushkar, involving admissible decompositions of fronts, has only been applied, to this author’s knowledge, to rederive results obtained through the Poincaré–Chekanov polynomial technique.) However, the Poincaré–Chekanov polynomial is not defined for all Legendrian knots, nor is it necessarily uniquely defined; in addition, as we shall see, there are many nonisotopic knots with the same polynomial. Higher-order Poincaré–Chekanov polynomials, which might also distinguish Legendrian knots, are difficult to compute and have not yet been successfully applied.

In Section 3.1, we introduce the characteristic algebra, a Legendrian invariant derived from the DGA, which is nontrivial, in all known cases, for Legendrian knots with maximal Thurston–Bennequin number. The characteristic algebra encodes the information from at least the first- and

second-order Poincaré–Chekanov polynomials, as we explain in Section 3.2. We will demonstrate the efficacy of our invariant, through examples, in Section 4.

Although the results of this section hold for links as well, we will confine our attention to knots for simplicity, except at the end of Section 3.1.

3.1. Definition of the characteristic algebra

The characteristic algebra can be viewed as a close relative of the DGA homology, except that it is easier to handle in general than the homology itself.

Definition 3.1. Let (A, ∂) be a DGA over $\mathbb{Z}[t, t^{-1}]$, where $A = \mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$, and let I denote the (two-sided) ideal in A generated by $\{\partial a_i \mid 1 \leq i \leq n\}$. The *characteristic algebra* $\mathcal{C}(A, \partial)$ is defined to be the algebra A/I , with grading induced from the grading on A .

Definition 3.2. Two characteristic algebras A_1/I_1 and A_2/I_2 are *tamely isomorphic* if we can add some number of generators to A_1 and the same generators to I_1 , and similarly for A_2 and I_2 , so that there is a tame isomorphism between A_1 and A_2 sending I_1 to I_2 .

In particular, tamely isomorphic characteristic algebras are isomorphic as algebras. Strictly speaking, Definition 3.2 only makes sense if we interpret the characteristic algebra as a pair (A, I) rather than as A/I , but we will be sloppy with our notation. Recall that we defined tame isomorphism between free algebras in Section 2.4.

A stabilization of (A, ∂) , as defined in Section 2.4, adds two generators e_1, e_2 to A and one generator e_2 to I ; thus A/I changes by adding one generator e_1 and no relations.

Definition 3.3. Two characteristic algebras A_1/I_1 and A_2/I_2 are *equivalent* if they are tamely isomorphic, after adding a (possibly different) finite number of generators (but no additional relations) to each.

Theorem 3.4. *Legendrian-isotopic knots have equivalent characteristic algebras.*

Proof. Let (A, ∂) be a DGA with $A = \mathbb{Z}[t, t^{-1}]\langle a_1, \dots, a_n \rangle$. Consider an elementary automorphism of A sending a_j to $a_j + v$, where v does not involve a_j ; since $\partial(a_j + v)$ is in I , it is easy to see that this automorphism descends to a map on characteristic algebras. We conclude that tamely isomorphic DGAs have tamely isomorphic characteristic algebras. On the other hand, equivalence of characteristic algebras is defined precisely to be preserved under stabilization of DGAs. \square

In the case of a link, we may also define the *link characteristic algebra* arising from the link DGA (A^L, ∂) introduced in Section 2.5, in a manner completely analogous to the definition of the characteristic algebra. Note that this gives a quotient of the path algebra A^L . Define equivalence of link characteristic algebras similarly to equivalence of characteristic algebras, except that replacing a generator a_i by $t_{u(a_i)}^{\pm 1} a_i$ or $t_{l(a_i)}^{\pm 1} a_i$ is allowed. Then Legendrian-isotopic links have equivalent link

characteristic algebras. An approach along these lines is used in [10] to distinguish between particular links.

3.2. Relation to the Poincaré–Chekanov polynomial invariants

In this section, we work over $\mathbb{Z}/2$ rather than over $\mathbb{Z}[t, t^{-1}]$; simply set $t = 1$ and reduce modulo 2. Thus we consider the DGA (A, ∂) of a Legendrian knot K over $\mathbb{Z}/2$, graded over $\mathbb{Z}/(2r(K))$; let $\mathcal{C} = A/I$ be its characteristic algebra.

We first review the definition of the Poincaré–Chekanov polynomials. The following term is taken from [5]:

Definition 3.5. Let (A, ∂) be a DGA over $\mathbb{Z}/2$. An algebra map $\varepsilon: A \rightarrow \mathbb{Z}/2$ is an *augmentation* if $\varepsilon(1) = 1$, $\varepsilon \circ \partial = 0$, and ε vanishes for any element in A of nonzero degree.

Given an augmentation ε of (A, ∂) , write $A_\varepsilon = \ker \varepsilon$; then ∂ maps $(A_\varepsilon)^n$ into itself for all n , and thus ∂ descends to a map $\partial^{(n)}: A_\varepsilon/A_\varepsilon^{n+1} \rightarrow A_\varepsilon/A_\varepsilon^{n+1}$. We can break $A_\varepsilon/A_\varepsilon^{n+1}$ into graded pieces $\sum_{i \in \mathbb{Z}/(2r(K))} C_i^{(n)}$, where $C_i^{(n)}$ denotes the piece of degree i . Write $\alpha_i^{(n)} = \dim_{\mathbb{Z}/2} \ker(\partial^{(n)}: C_i^{(n)} \rightarrow C_{i-1}^{(n)})$ and $\beta_i^{(n)} = \dim_{\mathbb{Z}/2} \text{im}(\partial^{(n)}: C_{i+1}^{(n)} \rightarrow C_i^{(n)})$, so that $\alpha_i^{(n)} - \beta_i^{(n)}$ is the dimension of the i th graded piece of the homology of $\partial^{(n)}$.

Definition 3.6. The *Poincaré–Chekanov polynomial of order n* associated to an augmentation ε of (A, ∂) is $P_{\varepsilon, n}(\lambda) = \sum_{i \in \mathbb{Z}/(2r(K))} (\alpha_i^{(n)} - \beta_i^{(n)}) \lambda^i$.

Note that augmentations of a DGA do not always exist.

The main result of this section states that we can recover some Poincaré–Chekanov polynomials from the characteristic algebra. To do this, we need one additional bit of information, besides the characteristic algebra.

Definition 3.7. Let γ_i be the number of generators of degree i of a DGA (A, ∂) graded over $\mathbb{Z}/(2r(K))$. Then the *degree distribution* $\gamma: \mathbb{Z}/(2r(K)) \rightarrow \mathbb{Z}_{\geq 0}$ of A is the map $i \mapsto \gamma_i$.

Clearly, the degree distribution can be immediately computed from a diagram of K by calculating the degrees of the vertices of K .

We are now ready for the main result of this section. Note that the following proposition uses the *isomorphism* class, not the equivalence class, of the characteristic algebra.

Proposition 3.8. *The set of first- and second-order Poincaré–Chekanov polynomials for all possible augmentations of a DGA (A, ∂) is determined by the isomorphism class of the characteristic algebra \mathcal{C} and the degree distribution of A .*

Before we can prove Proposition 3.8, we need to establish a few ancillary results. Our starting point is the observation that there is a one-to-one correspondence between augmentations and maximal ideals $\langle a_1 + c_1, \dots, a_n + c_n \rangle \subset A$ containing I and satisfying $c_i = 0$ if $\deg a_i \neq 0$.

Fix an augmentation ε . We first assume for convenience that $\varepsilon = 0$; then $I \subset M$, where M is the maximal ideal $\langle a_1, \dots, a_n \rangle$. For each i , write

$$\partial a_i = \partial_1 a_i + \partial_2 a_i + \partial_3 a_i,$$

where $\partial_1 a_i$ is linear in the a_j , $\partial_2 a_i$ is quadratic in the a_j , and $\partial_3 a_i$ contains terms of third or higher order. The following lemma writes ∂_1 in a standard form:

Lemma 3.9. *After applying a tame automorphism, we can relabel the a_i as $a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_{n-2k}$ for some k , so that $\partial_1 a_i = b_i$ and $\partial_1 b_i = \partial_1 c_i = 0$ for all i .*

Proof. For clarity, we first relabel the a_i as \tilde{a}_i . We may assume that the \tilde{a}_i are ordered so that $\partial \tilde{a}_i$ contains only terms involving \tilde{a}_j , $j < i$; see [1]. Let i_1 be the smallest number so that $\partial_1 \tilde{a}_{i_1} \neq 0$. We can write $\partial_1 \tilde{a}_{i_1} = \tilde{a}_{j_1} + v_1$, where $j_1 < i_1$ and the expression v_1 does not involve \tilde{a}_{j_1} . After applying the elementary isomorphism $\tilde{a}_{j_1} \mapsto \tilde{a}_{j_1} + v_1$, we may assume that $v_1 = 0$ and $\partial_1 \tilde{a}_{i_1} = \tilde{a}_{j_1}$.

For any \tilde{a}_i such that $\partial_1 \tilde{a}_i$ involves \tilde{a}_{j_1} , replace \tilde{a}_i by $\tilde{a}_i + \tilde{a}_{i_1}$. Then $\partial_1 \tilde{a}_i$ does not involve \tilde{a}_{j_1} unless $i = i_1$; in addition, no $\partial_1 \tilde{a}_i$ can involve \tilde{a}_{i_1} , since then $\partial_1^2 \tilde{a}_i$ would involve \tilde{a}_{j_1} . Set $a_1 = \tilde{a}_{i_1}$ and $b_1 = \tilde{a}_{j_1}$; then $\partial_1 a_1 = b_1$ and $\partial_1 \tilde{a}_i$ does not involve a_1 or b_1 for any other i .

Repeat this process with the next smallest \tilde{a}_{i_2} with $\partial_1 \tilde{a}_{i_2} \neq 0$, and so forth. At the conclusion of this inductive process, we obtain $a_1, \dots, a_k, b_1, \dots, b_k$ with $\partial_1 a_i = b_i$ (and $\partial_1 b_i = 0$), and the remaining \tilde{a}_i satisfy $\partial_1 \tilde{a}_i = 0$; relabel these remaining generators with c 's. \square

Now assume that we have relabelled the generators of A in accordance with Lemma 3.9.

Lemma 3.10. $\beta_\ell^{(1)}$ is the number of b_j of degree ℓ , while $\beta_\ell^{(2)} - \beta_\ell^{(1)}$ is the dimension of the degree ℓ subspace of the vector space generated by

$$\{\partial_2 b_i, \partial_2 c_i, a_i b_j + b_i a_j, b_i b_j, b_i c_j, c_i b_j\},$$

where i, j range over all possible indices.

Proof. The statement for $\beta_\ell^{(1)}$ is obvious. To calculate $\beta_\ell^{(2)} - \beta_\ell^{(1)}$, note that the image of $\partial^{(2)}$ in A/A^3 is generated by $\partial a_i = b_i + \partial_2 a_i$, $\partial b_i = \partial_2 b_i$, $\partial c_i = \partial_2 c_i$, $\partial(a_i a_j) = a_i b_j + b_i a_j$, $\partial(a_i b_j) = b_i b_j$, $\partial(b_i a_j) = b_i b_j$, $\partial(a_i c_j) = b_i c_j$, and $\partial(c_i a_j) = c_i b_j$. \square

We wish to write $\beta_\ell^{(n)}$ in terms of \mathcal{C} , but we first pass through an intermediate step. Let $N^{(n)}$ be the image of I in M/M^{n+1} , and let $\delta_\ell^{(n)}$ be the dimension of the degree ℓ part of $N^{(n)}$. Lemma 3.12 below relates $\beta_\ell^{(n)}$ to $\delta_\ell^{(n)}$ for $n = 1, 2$.

Lemma 3.11. $\delta_\ell^{(1)}$ is the number of b_i of degree ℓ , while $\delta_\ell^{(2)} - \delta_\ell^{(1)}$ is the dimension of the degree ℓ subspace of the vector space generated by

$$\{\partial_2 b_i, \partial_2 c_i, a_i b_j, b_i a_j, b_i b_j, b_i c_j, c_i b_j\},$$

where i, j range over all possible indices.

Proof. This follows immediately from the fact that I is generated by $\{\partial a_i, \partial b_i, \partial c_i\}$. \square

Lemma 3.12. $\beta_\ell^{(1)} = \delta_\ell^{(1)}$ and $\beta_\ell^{(2)} = \delta_\ell^{(2)} - \sum_{\ell'} \delta_{\ell'}^{(1)} \delta_{\ell-\ell'-1}^{(1)}$.

Proof. We use Lemmas 3.10 and 3.11. The first equality is obvious. For the second equality, we claim that, for fixed i and j , $a_i b_j$ only appears in conjunction with $b_i a_j$ in the expressions $\partial_2 b_m$ and $\partial_2 c_m$, for arbitrary m . It then follows that $\delta_\ell^{(2)} - \beta_\ell^{(2)}$ is the number of $a_i b_j$ of degree ℓ , which is $\sum_{\ell'} \delta_{\ell'}^{(1)} \delta_{\ell-\ell'-1}^{(1)}$.

To prove the claim, suppose that $\partial_2 b_m$ contains a term $a_i b_j$. Since $\partial_2^2 b_m = 0$ and $\partial_2(a_i b_j) = b_i b_j$, there must be another term in $\partial_2 b_m$ which, when we apply ∂_2 , gives $b_i b_j$; but this term can only be $b_i a_j$. The same argument obviously holds for $\partial_2 c_m$. \square

Now let ε be any augmentation, and let $M_\varepsilon = \langle a_1 + \varepsilon(a_1), \dots, a_n + \varepsilon(a_n) \rangle$ be the corresponding maximal ideal in A . If we define $N^{(n)}$ and $\delta_\ell^{(n)}$ as above, except with M replaced by M_ε , then Lemma 3.12 still holds. We are now ready to prove Proposition 3.8.

Proof of Proposition 3.8. Note that

$$(M_\varepsilon/M_\varepsilon^{n+1})/N^{(n)} \cong (M_\varepsilon/I)/(M_\varepsilon/I)^{n+1};$$

the characteristic algebra $\mathcal{C} = A/I$ and the choice of augmentation ε determine the right-hand side. On the other hand, the dimension of the degree ℓ part of $M_\varepsilon/M_\varepsilon^{n+1}$ is γ_ℓ if $n = 1$, and $\gamma_\ell + \sum_{\ell'} \gamma_{\ell'} \gamma_{\ell-\ell'}$ if $n = 2$. It follows that we can calculate $\{\delta_\ell^{(1)}\}$ and $\{\delta_\ell^{(2)}\}$ from \mathcal{C} , ε , and γ .

Fix $n = 1, 2$. By Lemma 3.12, we can then calculate $\{\beta_\ell^{(n)}\}$ and hence the Poincaré–Chekanov polynomial

$$P_{\varepsilon,n}(\lambda) = \sum_{\ell} ((\alpha_\ell^{(n)} + \beta_{\ell-1}^{(n)}) - \beta_\ell^{(n)} - \beta_{\ell-1}^{(n)}) \lambda^\ell.$$

Letting ε vary over all possible augmentations yields the proposition. \square

The situation for higher-order Poincaré–Chekanov polynomials seems more difficult; we tentatively make the following conjecture:

Conjecture 3.13. *The isomorphism class of \mathcal{C} and the degree distribution of A determine the Poincaré–Chekanov polynomials in all orders.*

Another set of invariants, similar to the Poincaré–Chekanov polynomials, are obtained by ignoring the grading of the DGA, and considering ungraded augmentations. In this case, the invariants are a set of integers, rather than polynomials, in each order. A proof similar to the one above shows that the first- and second-order ungraded invariants are determined by the characteristic algebra.

In practice, we apply Proposition 3.8 as follows. Given two DGAs, stabilize each with the appropriate number and degrees of stabilizations so that the two resulting DGAs have the same degree distribution. If these new DGAs have isomorphic characteristic algebras, then they have the same first- and second-order Poincaré–Chekanov polynomials (if augmentations exist). If not, then we can often see that their characteristic algebras are not equivalent, and so the original DGAs are not equivalent. Thus, calculating characteristic algebras often obviates the need to calculate first- and second-order Poincaré–Chekanov polynomials.

Note that the *first-order* Poincaré–Chekanov polynomials depend only on the *abelianization* of (A, ∂) ; if the procedure described above yields two characteristic algebras whose abelianizations are isomorphic, then the original DGAs have the same first-order Poincaré–Chekanov polynomials. On a related note, empirical evidence leads us to propose the following conjecture, which would yield a new topological knot invariant:

Conjecture 3.14. *For a Legendrian knot K with maximal Thurston–Bennequin number, the equivalence class of the abelianized characteristic algebra of K , considered without grading and over \mathbb{Z} , depends only on the topological class of K .*

Here the abelianization is unsigned: $vw = wv$ for all v, w .

We can view the abelianization of \mathcal{C} in terms of algebraic geometry. If $\mathcal{C} = (\mathbb{Z}/2)\langle a_1, \dots, a_n \rangle / I$, then the abelianization of \mathcal{C} gives rise to a scheme X in \mathbb{A}^n , affine n -space over $\mathbb{Z}/2$. Theorem 3.4 immediately implies the following result:

Corollary 3.15. *The scheme X is a Legendrian-isotopy invariant, up to changes of coordinates and additions of extra coordinates (i.e., we can replace $X \subset \mathbb{A}^n$ by $X \times \mathbb{A} \subset \mathbb{A}^{n+1}$).*

There is a conjecture about first-order Poincaré–Chekanov polynomials, suggested by Chekanov, which has a nice interpretation in our scheme picture.

Conjecture 3.16 (Chekanov [1]). *The first-order Poincaré–Chekanov polynomial is independent of the augmentation ε .*

Augmentations are simply the $(\mathbb{Z}/2)$ -rational points in X , graded in the sense that all coordinates corresponding to nonzero-degree a_j are zero. It is not hard to see that the first-order Poincaré–Chekanov polynomial at a $(\mathbb{Z}/2)$ -rational point p in X is precisely the “graded” codimension in \mathbb{A}^n of $T_p X$, the tangent space to X at p . The following conjecture, which we have verified in many examples, would imply Conjecture 3.16.

Conjecture 3.17. *The scheme X is irreducible and smooth at each $(\mathbb{Z}/2)$ -rational point.*

4. Applications

In this section, we give several illustrations of the constructions and results from Sections 2 and 3, especially Theorems 2.21 and 3.4. The first three examples, all knots, both illustrate the computation of the characteristic algebra described in Section 3.1, and demonstrate its usefulness in distinguishing between Legendrian knots. The last two examples, multi-component links, apply the techniques of Section 2.5 to conclude results about Legendrian links.

Instead of using the full DGA over $\mathbb{Z}[t, t^{-1}]$ or $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$, we will work over $\mathbb{Z}/2$ by setting $t = 1$ and reducing modulo 2. We remark that the characteristic algebra manipulations in this section, originally performed by hand, may seem ad hoc, but can be somewhat mechanized by computing noncommutative Gröbner bases, which can be done by computer.

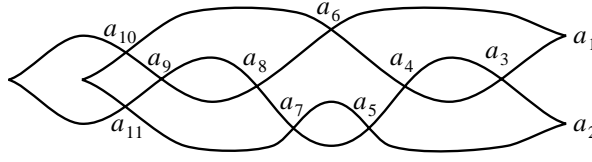


Fig. 9. The Legendrian knot K , of type 6_2 , with vertices labelled.

4.1. Example 1: 6_2

Our first example answers the Legendrian mirror question of Fuchs and Tabachnikov [9]; see also [12]. Let the *Legendrian mirror* of a Legendrian knot in \mathbb{R}^3 be the image of the knot under the involution $(x, y, z) \mapsto (x, -y, -z)$. It is asked in [9] whether a Legendrian knot with $r = 0$ must always be Legendrian isotopic to its mirror. We show that the answer is negative by using the characteristic algebra. Our proof is essentially identical to, but slightly cleaner than, the one given in [12]; rather than using the characteristic algebra, Ref. [12] performs an explicit computation on the DGA homology.

Let K be the unoriented Legendrian knot given in Fig. 9, which is of knot type 6_2 , with $r = 0$ and $tb = -7$. With vertices labelled as in Fig. 9, the differential on the DGA (A, ∂) for K is given by $A = \mathbb{Z}\langle a_1, \dots, a_{11} \rangle$ and

$$\begin{aligned} \partial a_1 &= 1 + a_{10}a_5a_3, & \partial a_6 &= a_{11}a_8, \\ \partial a_2 &= 1 + a_3(1 + a_6a_{10} + a_{11}a_7), & \partial a_7 &= a_8a_{10}, \\ \partial a_4 &= a_{11} + (1 + a_6a_{10} + a_{11}a_7)a_5, & \partial a_9 &= 1 + a_{10}a_{11}, \\ \partial a_5 &= \partial a_8 = \partial a_{10} = \partial a_{11} = \partial a_3 = 0. \end{aligned}$$

The ideal I is generated by the above expressions; the characteristic algebra of K is $\mathcal{C} = A/I$. The grading on A and \mathcal{C} is as follows: $a_1, a_2, a_7, a_9, a_{10}$ have degree 1; a_3, a_4 have degree 0; and a_5, a_6, a_8, a_{11} have degree -1 .

The characteristic algebra for the Legendrian mirror of K is the same as $\mathcal{C} = A/I$, but with each term in I reversed.

Lemma 4.1. *We have*

$$\begin{aligned} \mathcal{C} &\cong (\mathbb{Z}/2)\langle a_1, \dots, a_7, a_9, a_{10} \rangle / \\ &\langle 1 + a_{10}a_5a_3, 1 + a_3a_{10}a_5, 1 + a_{10}^2a_5^2, 1 + a_{10}a_5 + a_6a_{10} + a_{10}a_5^2a_7 \rangle. \end{aligned}$$

Proof. We perform a series of computations in $\mathcal{C} = A/I$:

$$\begin{aligned} a_8 &= a_8 + (1 + a_{10}a_{11})a_8 = a_{10}(a_{11}a_8) = 0; \\ 1 + a_6a_{10} + a_{11}a_7 &= a_{10}a_5a_3(1 + a_6a_{10} + a_{11}a_7) = a_{10}a_5; \\ a_{11} &= (1 + a_6a_{10} + a_{11}a_7)a_5 = a_{10}a_5^2. \end{aligned}$$

Substituting for a_8 and a_{11} in the relations $\{\partial a_i = 0\}$ yields the relations in the statement of the lemma. Conversely, given the relations in the statement of the lemma, and setting $a_8 = 0$ and $a_{11} = a_{10}a_5^2$, we can recover the relations $\{\partial a_i = 0\}$. \square

Decompose \mathcal{C} into graded pieces $\mathcal{C} = \oplus_i \mathcal{C}_i$, where \mathcal{C}_i is the submodule of degree i .

Lemma 4.2. *There do not exist $v \in \mathcal{C}_{-1}, w \in \mathcal{C}_1$ such that $vw = 1 \in \mathcal{C}$.*

Proof. Suppose otherwise, and consider the algebra \mathcal{C}' obtained from \mathcal{C} by setting $a_3 = 1, a_1 = a_2 = a_4 = a_6 = a_7 = a_9 = 0$. There is an obvious projection from \mathcal{C} to \mathcal{C}' which is an algebra map; under this projection, v, w map to $v' \in \mathcal{C}'_{-1}, w' \in \mathcal{C}'_1$, with $v'w' = 1$ in \mathcal{C}' . But it is easy to see that $\mathcal{C}' = (\mathbb{Z}/2)\langle a_5, a_{10} \rangle / \langle 1 + a_{10}a_5 \rangle$, with $a_5 \in \mathcal{C}'_{-1}$ and $a_{10} \in \mathcal{C}'_1$, and it follows that there do not exist such v', w' . \square

Proposition 4.3. *K is not Legendrian isotopic to its Legendrian mirror.*

Proof. Let $\tilde{\mathcal{C}}$ be the characteristic algebra of the Legendrian mirror of K . Since the relations in $\tilde{\mathcal{C}}$ are precisely the relations in \mathcal{C} reversed, Lemma 4.2 implies that there do not exist $v \in \tilde{\mathcal{C}}_1, w \in \tilde{\mathcal{C}}_{-1}$ such that $vw = 1$. On the other hand, there certainly exist $v \in \mathcal{C}_1, w \in \mathcal{C}_{-1}$ such that $vw = 1$; for instance, take $v = a_{10}$ and $w = a_5a_3$. Hence \mathcal{C} and $\tilde{\mathcal{C}}$ are not isomorphic. This argument still holds if some number of generators is added to \mathcal{C} and $\tilde{\mathcal{C}}$, and so \mathcal{C} and $\tilde{\mathcal{C}}$ are not equivalent. The result follows from Theorem 3.4. \square

More generally, the characteristic algebra technique seems to be an effective way to distinguish between some knots and their Legendrian mirrors; cf. Section 4.2. Note that Poincaré–Chekanov polynomials of any order can never tell between a knot and its mirror, since, as mentioned above, the differential for a mirror is the differential for the knot, with each monomial reversed.

4.2. Example 2: 7_4

Our second example shows that the characteristic algebra is effective even when Poincaré–Chekanov polynomials do not exist. In addition, this section and the next provide the first examples, known to the author, in which the DGA grading is not needed to distinguish between knots.

Consider the Legendrian knots K_1, K_2 shown in Fig. 10; both are of smooth type 7_4 , with $r = 0$ and $tb = 1$. We will show that K_1 and K_2 are not Legendrian isotopic.

The differential on the DGA for K_1 is given by

$$\begin{aligned} \partial a_1 &= 1 + a_8 a_{13} a_3, & \partial a_6 &= (1 + a_7 a_8) a_{13}, \\ \partial a_2 &= 1 + a_3 a_{13} a_8, & \partial a_9 &= a_{13} a_{11} + a_{10} a_{13}, \\ \partial a_4 &= a_5 a_8 a_{13} + a_{13} a_8 a_6, & \partial a_{10} &= 1 + a_{13} a_{12}, \\ \partial a_5 &= a_{13} (1 + a_8 a_7), & \partial a_{11} &= 1 + a_{12} a_{13}; \end{aligned}$$

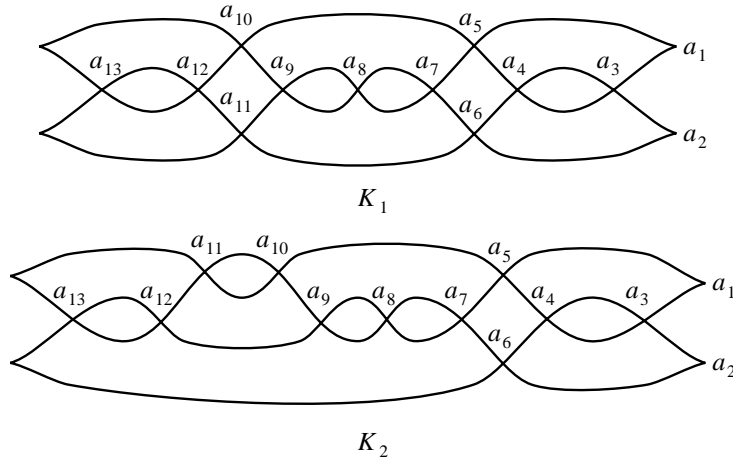


Fig. 10. The Legendrian knots K_1 and K_2 , of type 74, with vertices labelled.

the differential for K_2 is given by

$$\begin{aligned}\partial a_1 &= 1 + (1 + a_8 a_9 + a_8 a_{13} + a_{12} a_{13} + a_8 a_9 a_{12} a_{13} + a_8 a_{10} a_{11} a_{13}) a_3, \\ \partial a_2 &= 1 + a_3 a_{13} a_8, \\ \partial a_4 &= a_{13} a_8 a_6 + a_{11} a_{13} + a_5 (1 + a_8 a_9 + a_8 a_{13} + a_{12} a_{13} + a_8 a_9 a_{12} a_{13} + a_8 a_{10} a_{11} a_{13}), \\ \partial a_5 &= a_{13} (1 + a_8 a_7), \\ \partial a_6 &= a_7 + a_7 a_{12} a_{13} + (1 + a_7 a_8) (a_9 + a_{13} + a_9 a_{12} a_{13} + a_{10} a_{11} a_{13}), \\ \partial a_9 &= a_{10} a_{13}, \\ \partial a_{11} &= 1 + a_{13} a_{12}.\end{aligned}$$

Denote the characteristic algebras of K_1 and K_2 by $\mathcal{C}_1 = A/I_1$ and $\mathcal{C}_2 = A/I_2$, respectively; here $A = (\mathbb{Z}/2)\langle a_1, \dots, a_{13} \rangle$, and I_1 and I_2 are generated by the respective expressions above.

Lemma 4.4. *We have*

$$\mathcal{C}_1 \cong (\mathbb{Z}/2)\langle a_1, \dots, a_5, a_8, a_9, a_{10}, a_{13} \rangle / \langle 1 + a_3 a_8 a_{13}, a_3 a_8 + a_8 a_3, a_3 a_{13} + a_{13} a_3, a_8 a_{13} + a_{13} a_8 \rangle.$$

Proof. Similar to the proof of Lemma 4.1. \square

Lemma 4.5. *There is no expression in \mathcal{C}_1 which is invertible from one side but not from the other.*

Proof. It is clear that the only expressions in \mathcal{C}_1 which are invertible from either side are products of some number of a_3 , a_8 , and a_{13} , with inverses of the same form. Since a_3, a_8, a_{13} all commute, the lemma follows. \square

Lemma 4.6. *In \mathcal{C}_2 , a_{13} is invertible from the right but not from the left.*

Proof. Since $a_{13}a_{12} = 1$, a_{13} is certainly invertible from the right. Now consider adding to \mathcal{C}_2 the relations $a_3 = 1$, $a_7 = a_{13}$, $a_8 = a_{12}$, and $a_i = 0$ for all i not previously mentioned. The resulting algebra is isomorphic to $(\mathbb{Z}/2)\langle a_{12}, a_{13} \rangle / \langle 1 + a_{13}a_{12} \rangle$, in which a_{13} is not invertible from the left. We conclude that a_{13} is not invertible from the left in \mathcal{C}_2 either, as desired. \square

Proposition 4.7. *The Legendrian knots K_1 and K_2 are not Legendrian isotopic.*

Proof. From Lemmas 4.5 and 4.6, \mathcal{C}_1 and \mathcal{C}_2 are not equivalent. \square

Although \mathcal{C}_1 and \mathcal{C}_2 are not equivalent, one may compute that their abelianizations are isomorphic; cf. Conjecture 3.14. It is also easy to check that K_1 and K_2 have no augmentations, and hence no Poincaré–Chekanov polynomials.

The computation from the proof of Lemma 4.6 also demonstrates that K_2 is not Legendrian isotopic to its Legendrian mirror; we may use the same argument as in Section 4.1, along with the fact that a_{12} and a_{13} have degrees 2 and -2 , respectively, in \mathcal{C}_2 . By contrast, we see from inspection that K_1 is the same as its Legendrian mirror.

4.3. Example 3: 6_3 and 7_2

In a manner entirely analogous to Section 4.2, we can prove that many other pairs of Legendrian knots are not Legendrian isotopic. For example, consider the knots in Fig. 11: K_3 and K_4 , of smooth type 6_3 , with $r = 1$ and $tb = -4$, and K_5 and K_6 , of smooth type 7_2 , with $r = 0$ and $tb = 1$.

Proposition 4.8. *K_3 and K_4 are not Legendrian isotopic; K_5 and K_6 are not Legendrian isotopic.*

The proof of Proposition 4.8, which involves computations on the characteristic algebra along the lines of Section 4.2, is omitted here, but can be found in [11]. The 6_3 examples are the first known examples of two knots with nonzero rotation number which have the same classical invariants but are not Legendrian isotopic. The first-order Poincaré–Chekanov polynomial fails to distinguish between either the 6_3 or the 7_2 knots; K_3 and K_4 have no augmentations, while K_5 and K_6 both have first-order polynomial $\lambda + 2$. In fact, it can be computed that K_5 and K_6 share the same second-order polynomial as well.

4.4. Example 4: triple of the unknot

In this section, we rederive a result of [10] by using the link grading from Section 2.5. Our proof is different from the ones in [10].

Definition 4.9 (Michatchev [10]). Given a Legendrian knot K , let the n -copy of K be the link consisting of K , along with $n - 1$ copies of K slightly perturbed in the transversal direction. In the front projection, the n -copy is simply n copies of the front of K , differing from each other by small shifts in the z direction. We will call the 2-copy and 3-copy the *double* and *triple*, respectively.

Let $L = (L_1, L_2, L_3)$ be the unoriented triple of the usual “flying-saucer” unknot; this is the unoriented version of the link shown in Fig. 8.

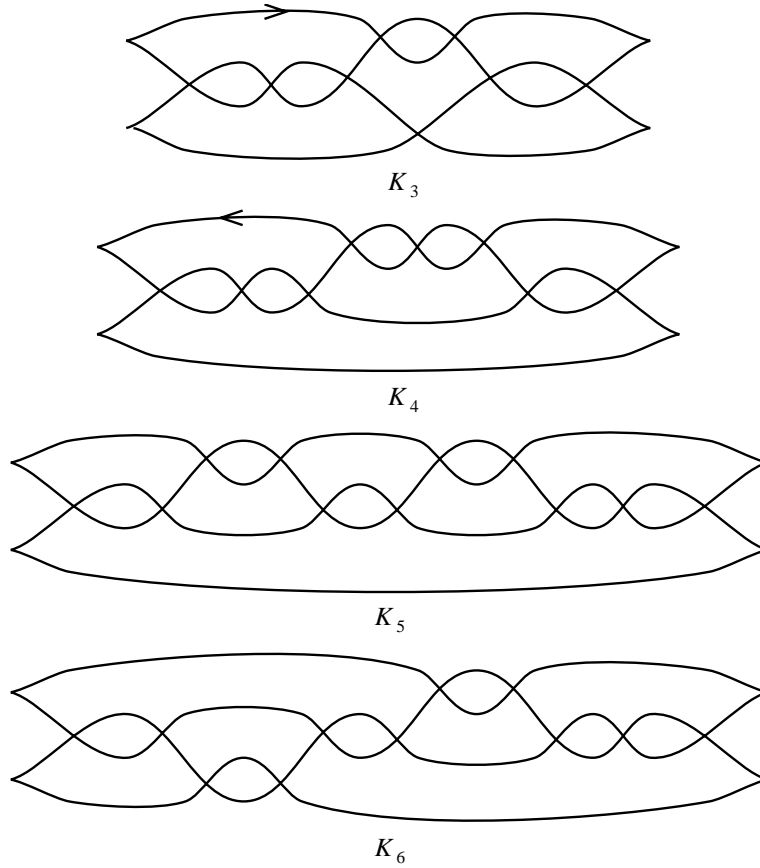


Fig. 11. The oriented Legendrian knots K_3 and K_4 , of type 63, and the unoriented knots K_5 and K_6 , of type 72.

Proposition 4.10 (Michatchev [10]). *The unoriented links (L_1, L_2, L_3) and (L_2, L_1, L_3) are not Legendrian isotopic.*

Proof. In Example 2.5, we have already calculated the first-order Poincaré–Chekanov polynomials for (L_1, L_2, L_3) , once we allow the grading (ρ_1, ρ_2) to range in $(\frac{1}{2}\mathbb{Z})^2$. The polynomials for the link (L_2, L_1, L_3) and grading $(\sigma_1, \sigma_2) \in (\frac{1}{2}\mathbb{Z})^2$ are identical, except with the indices 1 and 2 reversed. It is easy to compute that there is no choice of $\rho_1, \rho_2, \sigma_1, \sigma_2$ for which these polynomials coincide with the polynomials for (L_1, L_2, L_3) given in Example 2.5. The result now follows from Theorem 2.21. \square

4.5. Example 5: other links

In this section, we give two examples of other links which can be distinguished using our techniques. The proofs, which are simple and can be found in [11], use the Poincaré–Chekanov polynomial and Theorem 2.21, as in Section 4.4.

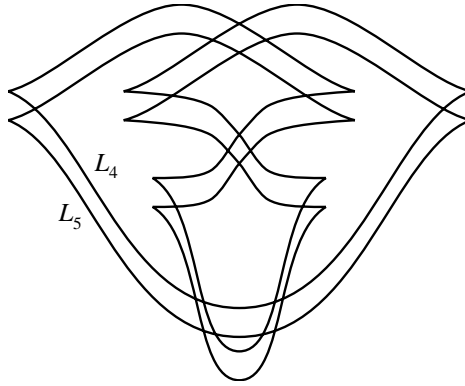


Fig. 12. The double of the figure eight knot from Fig. 4.

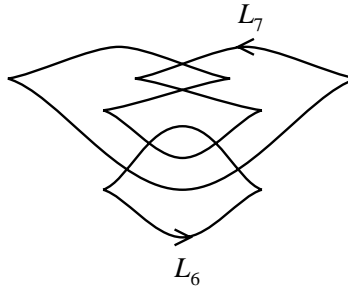


Fig. 13. The oriented Whitehead link.

Let (L_4, L_5) be the unoriented double of the figure eight knot, shown in Fig. 12, and let (L_5, L_4) be the same link, but with components interchanged. The following result answers a question from [10] about whether there is an unoriented knot whose double is not isotopic to itself with components interchanged.

Proposition 4.11. *The unoriented links (L_4, L_5) and (L_5, L_4) are not Legendrian isotopic.*

Our other example, in which orientation is important, is the oriented Whitehead link (L_6, L_7) shown in Fig. 13. Let $-L_j$ denote L_j with reversed orientation. By playing with the diagrams, one can show that (L_6, L_7) , $(L_7, -L_6)$, $(-L_6, -L_7)$, and $(-L_7, L_6)$ are Legendrian isotopic, as are $(-L_6, L_7)$, $(-L_7, -L_6)$, $(L_6, -L_7)$, and $(-L_7, -L_6)$. It is also the case that these two families are smoothly isotopic to each other. By contrast, we have the following result.

Proposition 4.12. *The oriented links (L_6, L_7) and $(-L_7, L_6)$ are not Legendrian isotopic.*

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